Resit / Numerical Mathematics 1 / July 11th 2022, University of Groningen

A simple calculator is allowed.
The exam is closed book.
All answers need to be justified using mathematical arguments.
Total time: 2 hours + 20 minutes (if special facilities were requested in advanced)

Exercise 1 (9 points)

Consider a function $f(x) \in C^2([a,b])$. We define the numerical integration rule:

$$\tilde{I}(f) = (b-a)f(\alpha)$$

with $\alpha \in [a,b]$.

(a) [3] Determine the degree of exactness for all possible values of α. Prove each of the results. Do not compute/use error bounds here.

For $\alpha = (a+b)/2$ we have the midpoint rule, and therefore it can be easily shown that linear functions can be integrated exactly **(0.5 pt)**, but not quadratic functions **(0.5 pt)**. For $\alpha \neq (a+b)/2$, constants can be integrated exactly **(1 pt)** but not linear functions **(1 pt)**

(b) [6] Prove an error bound for

$$\tilde{I}(f) - \int_a^b f(u)du$$

for all values of α. The error bound needs to depend on a, b, α and some derivative(s) of the function f. How does this bound compare with your results in the question before?

First, we need to do a Taylor expansion for approximating the the function which results in **(1 pt)**:

$$f(u) = f(\alpha) + (u-\alpha)f'(\alpha) + \frac{(u-\alpha)^2}{2}f''(\eta)$$

and therefore replacing and computing the integral **(1.5 pt)**:

$$\int_a^b f(u)du = f(\alpha)(b-a) + f'(\alpha)(b^2/2 - \alpha b - a^2/2 + \alpha a) + f''(\eta)\int_a^b \frac{(u-\alpha)^2}{2}$$

Note that for $\alpha = (b+a)/2$, the first term is zero, leading to the error bound **(1 pt)**:

$$\frac{(b-a)^3}{24} \max_{\eta \in [a,b]} |f''(\eta)|$$

Therefore, this result is consistent with the computation of degree of exactness since it means that linear functions are integrated exactly **(0.5 pt)**
For other values of α, the first term is non-zero. Therefore, the error bound should be re-written using Taylor theorem with order 1, leading to the error bound **(1 pt)**

$$\max_{\xi \in [a,b]} |f'(\xi)|((b-\alpha)^2/2 - (a-\alpha)^2/2)$$

Therefore, this result is consistent with the computation of degree of exactness since it means that constant functions can be integrated exactly **(1 pt)**.

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Exercise 2 (9 points)

Consider the root-finding problems for the functions $f(x) = \cos(x + \pi/2)$ and $g(x) = \cos(x) - 1$. Both have roots at $x = 0$.

(a) [4] Compute the Newton iteration function $f(x)$, and compute the first three derivatives of the iteration function

$$\phi_f(x) = x - \frac{\sin(x)}{\cos(x)} \quad \textbf{(1 pt)}$$

$$\phi_f'(x) = 1 - \frac{\cos(x)}{\cos(x)} - \frac{\sin^2(x)}{\cos^2(x)} = -\tan^2(x) \quad \textbf{(1 pt)}$$

$$\phi_f''(x) = -2\frac{\tan(x)}{\cos^2(x)} \quad \textbf{(1 pt)}$$

$$\phi_f'''(x) = -2\frac{1}{\cos^4(x)} - 4\frac{\tan(x)}{\cos^3(x)}\sin(x) \quad \textbf{(1 pt)}$$

(b) [3] Using Taylor expansions of the Newton iteration function ϕ_f of the function f at the root, determine the order of convergence for the Newton iterations $x^{(k)} = \phi_f(x^{(k-1)})$.

First, note that $\phi_f'(0) = \phi_f''(0) = 0$ **(1 pt)**, but $\phi_f''(0) \neq 0$ **(1 pt)**. Doing a Taylor expansion of $x^{(k)}$ around 0 **(1 pt)**:

$$x^{(k+1)} = \phi_f(x^{(k)}) = \phi_f(0) + \phi_f'(0)(x^{(k)}) + \phi_f''(0)(x^{(k)})^2/2 + \phi_f'''(\eta)(x^{(k)})^3/6 = \phi_f'''(\eta)(x^{(k)})^3/6$$

hence the order of convergence to zero is 3 **(1 pt)**.

(c) [2] Compute the Newton iteration function $g(x)$. What is the order of convergence to the root of these iterations, and why?

$$\phi_g(x) = x + \frac{\cos(x) - 1}{\sin(x)} \quad \textbf{(1 pt)}$$

Since the derivative of $g(x)$ is zero at the root, the order of convergence will be just linear **(1 pt)**.

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Exercise 3 (9 points)

Consider the ODE for the functions $y(t)$:

$$y' = f(y,t), y(0) = y_0$$

We define a numerical approximation for this ODE as follows:

$$y_{n+1} = y_n + \frac{h}{2}\left(f(y_n,t_n) + f(y_n + hf(y_n,t_n), t_{n+1})\right) \tag{1}$$

with $t_{n+1} = t_n + h$, $h > 0$.

(a) [1.5] Write down the formula for the Newton iterations for finding y_{n+1} in Equation (1).

The method is explicit, so that formula for the Newton method is the same equation. **(1.5 pt)**

(b) [4] Assume $f(y,t) = ay$, $a < 0$. Find h_{crit} such that $y_n \to 0$ for $h < h_{crit}$.

First **(1.5 pt)**

$$y_{n+1} = y_n + \frac{h}{2}(ay_n + a(y_n + hay_n)) = (1 + ah + (ah)^2/2)y_n$$

For A-stability, we then need **(1 pt)**:

$$(1 + ah + (ah)^2/2) < 1$$

what is equivalent to **(1.5 pt)**:

$$1 + ah/2 < 0 \Leftrightarrow h < -2/a = h_{crit}$$

(c) [3.5] It can be shown that for the method presented above it holds:

$$|y(t_{n+1}) - y_{n+1}| = Ch^3, C > 0.$$

List the advantages and disadvantages of method (1) compared to Forward Euler and Midpoint methods.

Compared to Forward Euler, this methods is second order accurate **(0.5 pt)** and and it is also explicit **(0.5 pt)**, and the stability limit for h is the same as for Forward euler **(0.5 pt)**. The increased order comes at the price of one additional evaluation of the function **(0.5 pt)**. Compared with Midpoint, both are second order accurate **(0.5 pt)** but Midpoint is implicit, therefore potentially more expensive **(0.5 pt)**, but midpoint is unconditionally stable for linear problems **(0.5 pt)**.